An Optimal Algorithm for Stochastic Three-Composite Optimization

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Three-Composite Convex Minimization

Consider the following convex three-composite problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} \left[P(\mathbf{x}) \triangleq f(\mathbf{x}) + g(\mathbf{x}) + h(\mathbf{A}\mathbf{x})\right]. \qquad \texttt{(P)}$$

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- ▷ Assume at least one minimizer $\mathbf{x}^*$ exists on $\mathbf{dom}P$.

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- Assume a stochastic first-order oracle $\mathsf{SFO}(f, \sigma)$ that returns an unbiased estimate of $\nabla f(\mathbf{x})$ with variance σ^2, for any $\mathbf{x} \in \mathbf{dom} P$.

Applications

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- ▷ Three-Composite Expected Risk Minimization
  - Graph-Guided Fused Lasso
  - Graph-Guided Sparse Logistic Regression
  - Robust Matrix Recovery

Saddle-Point Reformulation

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^d} \; f(\mathbf{x}) + g(\mathbf{x}) + h(\mathbf{A}\mathbf{x})}$$

$\mathbf{prox}_{h\circ\mathbf{A}}$ is intractable in general $\Rightarrow$ Consider saddle-point form

$$\min_{\mathbf{x}\in\mathbb{R}^d} \max_{\mathbf{y}\in\mathbb{R}^m} \left[S(\mathbf{x},\mathbf{y}) \triangleq f(\mathbf{x}) + g(\mathbf{x}) + \langle \mathbf{A}\mathbf{x}, \mathbf{y}\rangle - h^*(\mathbf{y}) \right]. \qquad (\texttt{SP})$$

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- Develop a primal-dual algorithm for ($\texttt{SP}$).

Existing methods

$$\min_{\mathbf{x}\in\mathbb{R}^d} \; f(\mathbf{x}) + g(\mathbf{x}) + h(\mathbf{A}\mathbf{x})$$

Algorithm	Reference	Type	Convergence Rate[1]	K Known?
Stoc. Subgradient	[Lan12]	Primal	$O\left(\frac{L}{K^2} + \frac{M_g + M_h B + \sigma}{\sqrt{K}}\right)$	Yes
Stoc. F-B Splitting	[DS09]	Primal	$O\left(\frac{\max\{L + M_h B, M_g\} + \sigma}{\sqrt{K}}\right)$	Yes
Stoc. ADMM	[Ouy13]	Primal-Dual	$O\left(\frac{L + M_g}{\sqrt{K}} + \frac{B}{K} + \frac{\sigma}{\sqrt{K}}\right)$	No
Stoc. E-ADMM	[Lin18]	Primal-Dual	$O\left(\frac{L}{K} + \frac{B}{K} + \frac{\sigma^2}{\sqrt{K}}\right)$	No
Stoc. NSPA	[ZK14]	Primal-Dual	$O\left(\frac{L}{K^2} + \frac{M_g^2}{K^{3/2}} + \frac{B^2}{K} + \frac{\sigma}{\sqrt{K}}\right)$	No
Stoc. PD3CM	[ZC18]	Primal-Dual	$O\left(\frac{L}{K} + \frac{B}{K} + \frac{\sigma}{\sqrt{K}}\right)$	Yes

L: Smoothness of f
B: Operator norm of $\mathbf{A}$
M_g: Lipschitz constant of g
M_h: Lipschitz constant of h
σ^2: Variance of stochastic (sub-)gradient
K: Total number of iterations

[1] In terms of expected primal sub-optimality gap or primal-dual gap.

Lower Bound of Convergence Rate

Under $\mathsf{SFO}(f, \sigma)$, when $g \equiv 0$, for any algorithm that solves (**SP**), the convergence rate is no better than[2]

$$\Omega\left(\frac{L}{K^2} + \frac{B}{K} + \frac{\sigma}{\sqrt{K}}\right). \tag{LB}$$

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Yes, we can!

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Choice of Parameters

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The convergence rate of our algorithm matches the lower bound (**LB**) for any values of ρ and ρ'.

Extension to Multi-Composite Problems

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) + g(\mathbf{x}) + \sum_{i=1}^{p} h_i(\mathbf{A}_i\mathbf{x})} \quad \Longrightarrow \text{Product-Space Technique}$$

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- ▷ For large p, can further introduce randomization on the dual update and averaging steps.

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A3 $\mathbb{E}_{\boldsymbol{\xi}^k}\left[\exp\left\{\varsigma\|\boldsymbol{\varepsilon}^k\|^2/\sigma^2\right\} \,\middle|\, \mathcal{F}_k\right] \le \exp\{\varsigma^2+\varsigma\}$ *a.s.*

An Important Lemma

Lemma 2 (Z.-Haskell-Tan, 2018)

Let **dom** g *be compact and* **dom** h^* *be bounded. In Algorithm I, let* $\beta_0 = 1$,

$$\beta_{k-1}\theta_k + 1 = \beta_k, \forall\, k \in \mathbb{Z}^+,$$
$$0 < \theta_k \leq \min\{\tau_{k-1}/\tau_k, \alpha_{k-1}/\alpha_k\}, \forall\, k \in \mathbb{N},$$
$$B^2\alpha_{k-1} + L/\beta_{k-1} \leq (1-\zeta)/\tau_{k-1}, \forall\, k \in \mathbb{N},$$

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for some $\zeta \in (0,1)$.

Define $\Gamma_K \triangleq \sum_{k=0}^{K-1} \gamma_k \tau_k$ *and* $\Gamma'_K \triangleq (\sum_{k=0}^{K-1} \gamma_k^2)^{1/2}$. *If* ***A1*** *and* ***A2*** *hold, then*

$$\mathbb{E}_{\Xi_K}\left[G(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K)\right] \le \frac{D_g^2}{\beta_{K-1}\tau_{K-1}} + \frac{D_{h^*}^2}{2\beta_{K-1}\alpha_{K-1}} + \frac{(1+\zeta)\Gamma_K}{2\zeta\beta_{K-1}\gamma_{K-1}}\sigma^2,\ \forall\, K \in \mathbb{N}.$$

An Important Lemma (Cont'd)

*Also, if **A1** and **A3** hold, then for any* $\delta \in (0,1)$*,*

$$G(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K) \leq \frac{1}{\beta_{K-1}} \left\{ \frac{4\sqrt{\log(2/\delta)}D_g}{\gamma_{K-1}} \Gamma'_K \sigma + \frac{D_g^2}{\tau_{K-1}} + \frac{D_{h^*}^2}{2\alpha_{K-1}} + \frac{1 + 2\sqrt{\log(2/\delta)}}{2\zeta\gamma_{K-1}(1+\zeta)^{-1}} \Gamma_K \sigma^2 \right\}$$

with probability (w.p.) at least $1 - \delta$*.*

Main Results

Theorem 3 (Z.-Haskell-Tan, 2018)

Let $\mathbf{dom}\, g$ *be compact,* $\mathbf{dom}\, h^*$ *be bounded and* $(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K)$ *be produced by Algorithm I. If* ***A1*** *and* ***A2*** *hold, then for any* $K \in \mathbb{N}$*,*

$$\mathbb{E}_{\Xi_K}\left[G(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K)\right] \leq \frac{8L}{K(K+3)} D_g^2 + \frac{4B}{K}\left(\rho' D_g^2 + \frac{D_{h^*}^2}{4\rho'}\right) + \frac{4\sigma}{\sqrt{K+3}}\left(\rho D_g^2 + \frac{2}{\rho}\right).$$

$$= O\left(\frac{L}{K^2} + \frac{B}{K} + \frac{\sigma}{\sqrt{K}}\right)$$

Main Results (Cont'd)

*In addition, if **A1** and **A3** hold, then for any* $\delta \in (0,1)$*,*

$$\begin{aligned} G(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K) &\leq \frac{8L}{K(K+3)} D_g^2 + \frac{4B}{K}\left(\rho' D_g^2 + \frac{D_{h^*}^2}{4\rho'}\right) \\ &\quad + \frac{16\sigma}{\sqrt{K+3}}\left(D_g + \frac{2}{\rho}\right)\sqrt{\log(2/\delta)} \\ &= O\left(\frac{L}{K^2} + \frac{B}{K} + \frac{\sigma\sqrt{\log(1/\delta)}}{\sqrt{K}}\right) \end{aligned} \tag{2}$$

w.p. at least $1-\delta$.

Choose ρ and ρ'

If D_g and D_{h^*} are known or can be estimated, then we can choose $\rho' = D_{h^*}/(2D_g)$ and $\rho = 2/D_g$. As a result,

$$\mathbb{E}_{\Xi_K}\left[G(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K)\right] \leq \frac{8L}{K(K+3)} D_g^2 + \frac{4B}{K} D_g D_{h^*} + \frac{12\sigma}{\sqrt{K+3}} D_g$$

and for any $\delta \in (0,1)$, w.p. at least $1-\delta$,

$$G(\overline{\mathbf{x}}^K, \overline{\mathbf{y}}^K) \leq \frac{8L}{K(K+3)} D_g^2 + \frac{4B}{K} D_g D_{h^*} + \frac{32\sigma}{\sqrt{K+3}} \sqrt{\log(2/\delta)} D_g.$$

Constrained Minimization Reformulation

$$\boxed{\min_{\mathbf{u}\in\mathbb{R}^d,\boldsymbol{\omega}\in\mathbb{R}^m} \; f(\mathbf{u}) + g(\mathbf{u}) + h(\boldsymbol{\omega}) \quad \text{s.t.} \quad \mathbf{A}\mathbf{u} = \boldsymbol{\omega}} \tag{\texttt{CSP}}$$

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When $g \equiv 0$:

- ▷ Many stochastic ADMM algorithms proposed [Ouy13; Suz13; AS14].
- ▷ The algorithm in [AS14] obtains the optimal convergence rate
  $\longrightarrow$ The convergence rate of the smooth term f is $O(L/K^2)$.

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When g is CCP:

- ▷ How to design an optimal ADMM algorithm for ($\texttt{CSP}$)?
- ▷ Moreover, how is it related to Algorithm I?

Algorithm II: A Stochastic ADMM Algorithm

- **Define**: $L_k^{\varrho}(\mathbf{u},\boldsymbol{\omega},\boldsymbol{\lambda}) \triangleq f(\mathbf{u}^k) + \langle \mathbf{v}^k, \mathbf{u}-\mathbf{u}^k\rangle + g(\mathbf{u}) + h(\boldsymbol{\omega}) - \langle \boldsymbol{\lambda}, \mathbf{A}\mathbf{u}-\boldsymbol{\omega}\rangle$
$+r_k(2\eta_k)^{-1}\langle \mathbf{u}-\mathbf{u}^k, \mathbf{W}^k(\mathbf{u}-\mathbf{u}^k)\rangle + (\varrho/2)\|\mathbf{A}\mathbf{u}-\boldsymbol{\omega}\|^2$

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Sample $\tilde{\boldsymbol{\xi}}^k \sim \nu$ and define $\tilde{\mathbf{v}}^k \triangleq \nabla_{\mathbf{u}} F(\mathbf{u}, \tilde{\boldsymbol{\xi}}^k)|_{\mathbf{u} = \tilde{\mathbf{u}}^k}$

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- **Initialize**: $\mathbf{u}^0 \in \mathbf{dom}\, g$, $\boldsymbol{\omega}^0 \in \mathbf{dom}\, h$, $\boldsymbol{\lambda}^0 \in \mathbb{R}^m$, $\overline{\mathbf{u}}^0 = \mathbf{u}^0$, $\overline{\boldsymbol{\omega}}^0 = \boldsymbol{\omega}^0$, $\overline{\boldsymbol{\lambda}}^0 = \boldsymbol{\lambda}^0$, $k = 0$
- **Repeat** (until some convergence criterion is met)

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Sample $\tilde{\boldsymbol{\xi}}^k \sim \nu$ and define $\tilde{\mathbf{v}}^k \triangleq \nabla_{\mathbf{u}} F(\mathbf{u}, \tilde{\boldsymbol{\xi}}^k)|_{\mathbf{u}=\tilde{\mathbf{u}}^k}$

$$\boldsymbol{\omega}^{k+1} := \operatorname*{arg\,min}_{\boldsymbol{\omega}\in\mathbf{dom}\, h} L_k^{\varrho}(\mathbf{u}^k, \boldsymbol{\omega}, \boldsymbol{\lambda}^k)$$

$$\mathbf{u}^{k+1} := \operatorname*{arg\,min}_{\mathbf{u}\in\mathbf{dom}\, g} L_k^{\varrho}(\mathbf{u}, \boldsymbol{\omega}^{k+1}, \boldsymbol{\lambda}^k) \qquad \text{(Alternating Update)}$$

Algorithm II: A Stochastic ADMM Algorithm

- **Define**: $L_k^\varrho(\mathbf{u},\boldsymbol{\omega},\boldsymbol{\lambda}) \triangleq f(\mathbf{u}^k) + \langle \mathbf{v}^k, \mathbf{u}-\mathbf{u}^k\rangle + g(\mathbf{u}) + h(\boldsymbol{\omega}) - \langle \boldsymbol{\lambda}, \mathbf{A}\mathbf{u}-\boldsymbol{\omega}\rangle$
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$$\boldsymbol{\lambda}^{k+1} := \boldsymbol{\lambda}^k - \varrho(\mathbf{A}\mathbf{u}^{k+1} - \boldsymbol{\omega}^{k+1}) \qquad \text{(Multiplier Update)}$$

Algorithm II: A Stochastic ADMM Algorithm (Cont'd)

$$(\overline{\boldsymbol{\omega}}^{k+1}, \overline{\mathbf{u}}^{k+1}, \overline{\boldsymbol{\lambda}}^{k+1}) := r_k(\boldsymbol{\omega}^{k+1}, \mathbf{u}^{k+1}, \boldsymbol{\lambda}^{k+1}) + (1 - r_k)(\overline{\boldsymbol{\omega}}^{k}, \overline{\mathbf{u}}^{k}, \overline{\boldsymbol{\lambda}}^{k}) \qquad \text{(Averging)}$$

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Using variable substitution and Moreau's identity,

Algorithm II is equivalent to Algorithm I with unit extrapolation parameter.

Numerical Experiments: Setup

	Abbrev.	Algorithms
Ours	OTPDHG	Algorithm I & Multi-Comp. Ext.
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Graph-Guided Fused Logistic Regression (GLR)

$$P_{\mathrm{GLR}}(\mathbf{x}) = n^{-1}\sum_{i=1}^{n} \log(1 + \exp(-b_i \mathbf{a}_i^T \mathbf{x})) + \lambda_1 \|\mathbf{x}\|_1 + \lambda_2 \|\mathbf{F}\mathbf{x}\|_1$$

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  - Let $\pi : \mathcal{E} \to [|\mathcal{E}|]$ be any bijection.
  - For any edge $(i,j) \in \mathcal{E}$ $(i < j)$, $\mathbf{F}_{\pi(i,j),i} = w(i,j)$, $\mathbf{F}_{\pi(i,j),j} = -w(i,j)$ and $\mathbf{F}_{\pi(i,j),s} = 0$ for all $s \in [d] \setminus \{i,j\}$.

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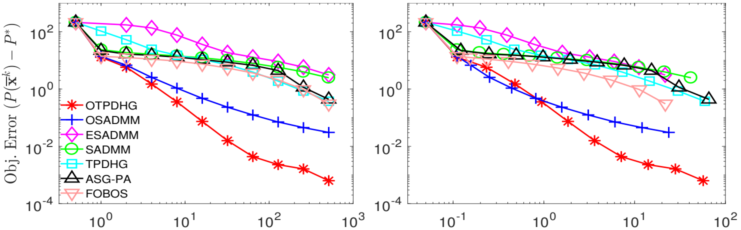
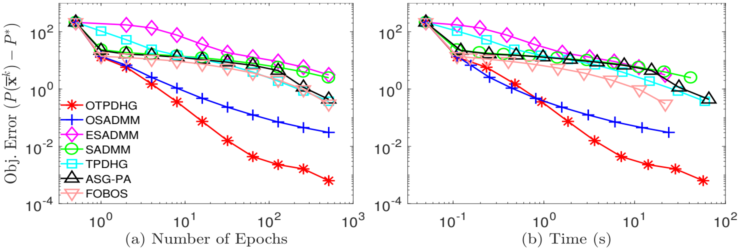
a9a



(a) Number of Epochs

(b) Time (s)

a9a


Obj. Error $(P(\overline{\mathbf{x}}^k) - P^*)$
OTPDHG
OSADMM
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(a) Number of Epochs
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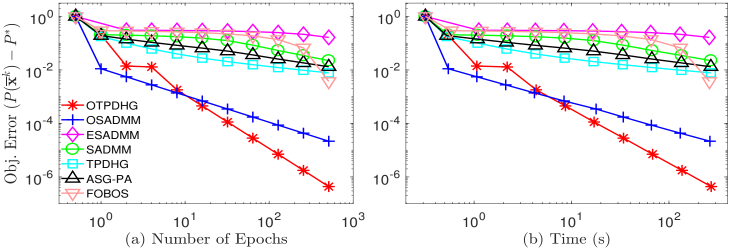
covtype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Sparse Overlapping Group Lasso (OGL)

$$P_{\mathrm{GLR}}(\mathbf{x}) = n^{-1}\sum_{i=1}^{n}(\mathbf{a}_i^T\mathbf{x} - b_i)^2/2 + \lambda_0\|\mathbf{x}\|_1 + \sum_{i=1}^{p}\lambda_i\|\mathbf{x}_{\mathcal{G}_i}\|$$

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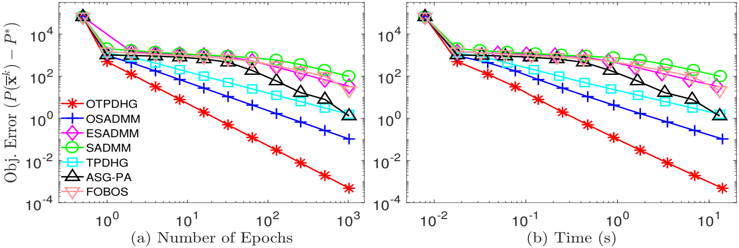
- $\mathbf{x}_{\mathcal{G}_i}$ denotes the subvector of $\mathbf{x}$ indexed by $\mathcal{G}_i \subseteq [d]$.
- Solved by multi-composite extensions of Algorithms I and II.

cpu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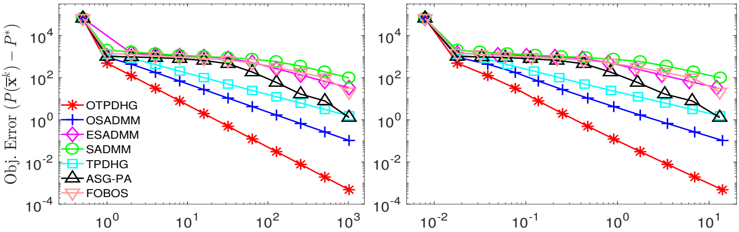

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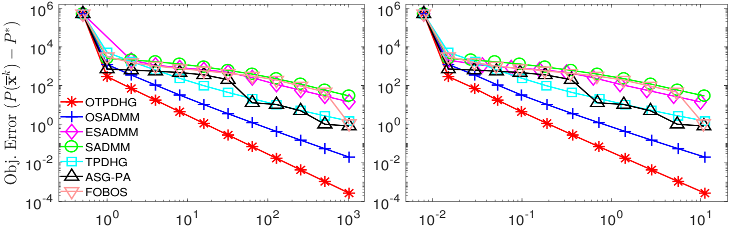
abalone



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Future work

- Consider strongly convex f.
- Extend to non-Euclidean geometry.
- Consider randomized matrix-vector product $\mathbf{Ax}$ and $\mathbf{A}^T\mathbf{y}$.
- Extend to non-bilinear structure.

Thank you!

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